Theorem 2.2

Suppose M_i is a cube with n_i handles, where $i \in \{1, 2\}$.

Then M_1 is homeomorphic to M_2 iff $n_1 = n_2$ and either both are orientable or both are non-orientable.

Proof. One direction is obvious.

Now suppose $n = n_1 = n_2$, and let

$$h_i : \bigcup_{j=1}^{n} D_{ij} \times [-1, 1] \to M_i$$

be such that $R_i = M_i \setminus h_i(\cup D_{ij} \times (-1,1))$

is a 3-cell.

Suppose the M_i's are both orientable.

This induces an orientation on R_i,

and hence on ∂R_i, a 2-manifold.

Now $h_i(D_{ij} \times \{\pm 1\}) \subseteq \partial R_i$,

and orientation of $h_i(D_{ij} \times \{1\})$ is

opposite from the one of

$h_i(D_{ij} \times \{-1\})$.

Theorem 1.5 gives us a homeomorphism $f: \partial R_1 \to \partial R_2$ taking

$h_1(D_{1j} \times \{\pm 1\})$ onto

$h_2(D_{2j} \times \{\pm 1\})$ in a way

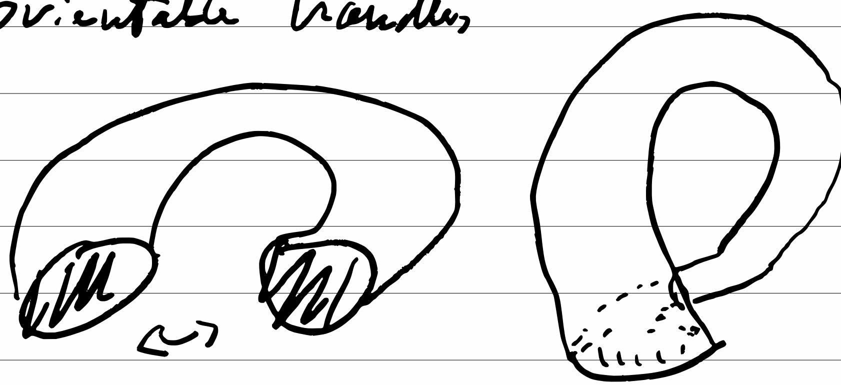
respecting the orientations.

Hence $f: \partial R_1 \to \partial R_2$ is orientation-preserving, and Thm 1.4 allows us to extend this to an orientation-preserving homeomorphism of $R_1 \to R_2$, which then easily extends to

$$M_1 \xrightarrow{\cong} M_2 .$$

Now the non-orientable case.

In this case we have $r_i \geq 1$ non-orientable handles

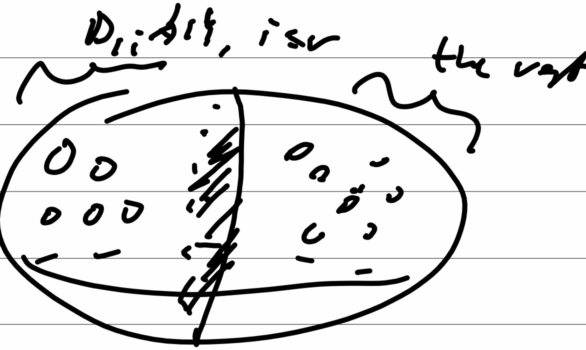


It seems that the number r_i should matter, but it doesn't.

Order the discs D_{1i} so that $D_{11} \times \{1\}, \dots, D_{1r_1} \times \{1\}$ have different orientations, then $D_{11} \times \{-1\}, \dots, D_{1r+1}$

but $D_{1,r+1} \times \{\pm 1\}, \dots, D_{1n} \times \{\pm 1\}$

have the same .



E_1

Pick a properly embedded disc E_1 in R_1 which separates $D_{1i} \times \{1\}$, $i \le r_1$, from the rest. Do the same in R_2 with E_2. Now cut M_i along $\{E_i, D_{i2}, \dots, D_{in}\}$.

The resulting cube with handles has exactly <u>one</u> non-orientable handle, corresponding to E_i.

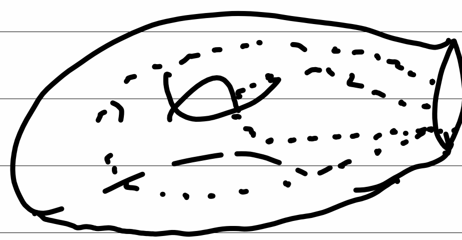
Now repeat the argument from the orientable case, taking $E_1 \times \{\pm 1\}$ to $E_2 \times \{\pm 1\}$, and $D_{1i} \times \{\pm 1\}$ to $D_{2i} \times \{\pm 1\}$ for $i \geq 2$.

<u>Theorem 2.3</u>

F is a compact, connected 2-manifold (surface) with $\partial F \neq \emptyset$.

Then $F \times I$, $I = [0,1]$, is a cube with n handles, where $n = 1 - \chi(F)$, and $F \times I$ is orientable iff F is.

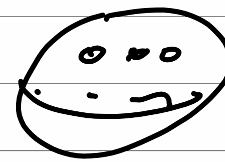
Proof



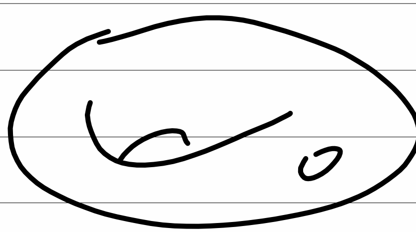
$\exists$ $n = 1 - \chi(F)$ pairwise disjoint properly embedded arcs (1-cells) $A_1, \dots, A_n$ which cut F into a disc (2-cell).

Now the discs $A_1 \times I, \ldots, A_k \times I$

cut $F \times I$ into $B^2 \times I \cong B^3$. $\square$

Example Take $F_1 =$ 

three-holed sphere,

$F_2 =$  one-holed torus.

Note $F_1 \not\cong F_2$ (but $F_1 \simeq F_2$),

and $\chi(F_1) = \chi(F_2) = -1$

Hence $F_1 \times I$, $F_2 \times I$ are cubes with

2 handles, and therefore are homeomorphic.

$F_1 \not\cong F_2$ but $F_1 \times I \cong F_2 \times I$

[stably homeomorphic]

Theorem 2.4

Γ connected, finite 1-complex in a 3-manifold M. Every regular n'hood of Γ in M is a cube with n handles, where $n = 1 - \chi(\Gamma)$.

<u>Proof</u> We pick a triangulation K of M which contains P as a subcomplex. Let $N = N(P, K'')$ be the regular n'hood we discussed before (union of stars).

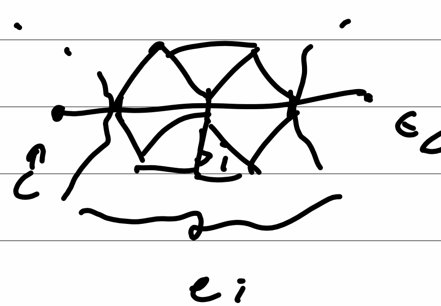
Let T be a maximal tree in P; let $e_1, \dots, e_n$ be edges of $P \setminus T$. There are precisely $n = 1 - \chi(P)$ of them.

Let $C = N(T, K'')$. By Corollary 1.2,

C is a 3-cell.

Let b_i be the barycenter of e_i.

$B_i = St(b_i, K'')$ is a 3-cell, since K'' is a combinatorial triangulation. Furthermore, $B_i \cap C$ is a disjoint union of two discs $D_{i,-1}$, $D_{i,1}$, each being a star of a vertex inside ∂C.



Thus there is a homeomorphism

$h_i : B^2 \times [-1,1] \to B_i$ with

$h_i(B^2 \times \{\pm 1\}) = D_{i,\pm 1}$.

The collection $\{h_i(B^2 \times \{0\})\}$

cuts N into the 3-cell C. $\square$

Splittings and diagrams

Def A Heegaard splitting of a

closed connected 3-manifold M

is a pair (V_1, V_2) of cubes

with handles, with $M = V_1 \cup V_2$ and

$\partial V_1 = \partial V_2 = V_1 \cap V_2$.

Note that if V_1 has n handles, then ∂V_1 is a closed surface of genus n, and Euler characteristic $2-2n$, which is orientable iff V is.

Thus, if (V_1, V_2) is a Heegaard splitting, then V_1 and V_2 have the same number of handles (called the <u>genus</u> of the splitting), and are either both orientable or both non-orientable.

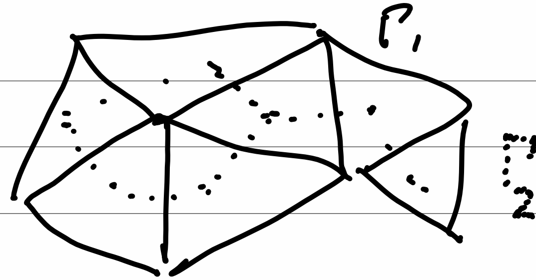
Theorem 2.5

Each closed, connected 3-mfld has a Heegaard splitting.

Proof

Let K be a triangulation of M.

Let Γ_1 be the 1-skeleton of K, and let Γ_2 be the dual 1-skeleton, i.e., Γ_2 is the maximal 1-dim subcomplex of K' not intersecting Γ_1.



Let $V_i = N(P_i, K'')$.

By Corollary 1.7, V_1 is a regular n'hood of P_1. V_2 is also, but one has to verify Thm 1.6 for this.

By Thm 2.4, both V_1 and V_2 are cubes with handles.

Now it is immediate that $V_1 \cup V_2 = M$

and that $\partial V_1 = \partial V_2 = V_1 \cap V_2$. $\square$